$$\frac{\partial f}{\partial x}(0.0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0.0)}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = 0$$

$$\frac{\partial f}{\partial x}(0.0) = \lim_{k \to 0} \frac{f(0,0+k) + f(0.0)}{k} = 0$$

Thus both partial derivatives $\frac{\partial f}{\partial x}(0.0)$ exist but f is not continuous at (0.0).

ex: $f(x,y) = x^2y$

$f_x = 2xy$, $f_y = x^2$

* Second order partial derivatives

$$f_{xx} = (f_x)_x = \frac{\partial f_x}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial f_x}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial f_y}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial f_y}{\partial y} = \frac{\partial^2 y}{\partial y \partial y}$$

ex: $f(x,y) = x^2y$

$f_x = 2xy$ $\quad$ $f_y = x^2$

$f_{xx} = 2y$ $\quad$ $f_{yx} = 2x$

$f_{xy} = 2x$ $\quad$ $f_{yy} = 0$

THEOREM: If $f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are continuous then

$$f_{xy} = f_{yx}$$

(book 72)

ex: $f(x,y) = \begin{cases} xy \dfrac{x^2-y^2}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$

$$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{\substack{k \to 0 \\ k \neq 0}} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f = \frac{x^3y - y^3x}{x^2+y^2}$$

$$\underset{k \neq 0}{f_x(0,k)} = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - y^3x)\,2x}{(x^2+y^2)^2}\Bigg|_{0}^{k}$$

$$= \frac{(-k^3)(k^2)}{k^4} = -k$$

$$\lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{-k - 0}{k} = -1$$

$$f_{yx}(0,0) = (f_y)_x(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f(0,0)}{h}$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$f_y(h,0) = \left. \frac{(x^3-y^3)(x^2+y^2) - (x^3y - y^3x)\cdot 2x}{(x^2+y^2)^2} \right|_{(h,0)}$$

$$= \frac{h^3 \cdot h^2}{h^4} = h$$

$$\lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h-0}{h} = 1$$

ex: $w = e^x + x\ln y + y\ln x$

Show $w_{xy} = w_{yx}$

$$w_x = e^x + \ln y + y \cdot \frac{1}{x}$$

$$w_{xy} = \frac{1}{y} + \frac{1}{x}$$

$$w_y = x \cdot \frac{1}{y} + \ln x$$

$$w_{yx} = \frac{1}{y} + \frac{1}{x}$$

14.4 → THE CHAIN RULE

* 1 variable

$w = f(x)$, $x = g(t)$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

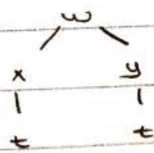
* 2 variable

$w = f(x,y)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$x = g(t)$

$y = h(t)$



ex: $w = xy$

$x = \cos t \qquad y = \sin t$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2}$$

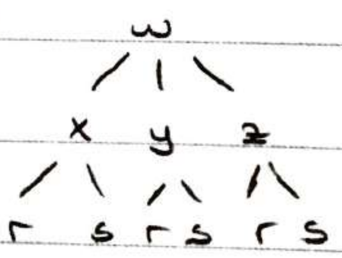
$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= y(-\sin t) + x \cos t$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = y(\pi/2) \cdot (-\sin \pi/2) + x(\pi/2) \cancel{\cos \pi/2}$$

$$= 1(-1) = -1$$

***



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

ex: $w = x + 2y + z^2$

$x = r/s \quad , \quad y = r^2 + \ln s \quad , \quad z = 2r \qquad \frac{\partial w}{\partial s} = ? \quad \frac{\partial w}{\partial r} = ?$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= 1 \cdot \left(\frac{-r}{s^2}\right) + 2 \cdot \frac{1}{s} + 2z \cdot 0$$

$$\frac{\partial w}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$\begin{tikzcd}[column sep=small] & w \arrow[d, no head] & \\ & x \arrow[dl, no head] \arrow[dr, no head] & \\ s & & r \end{tikzcd}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial s}$$

$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r}$$

Implicit Differentiation

* $F(x,y) = 0$

$$y = f(x)$$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} 0 = 0$$

$$F_x \frac{dx}{dy} + F_y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y}$$

* $F(x,y,z) = 0$

$$z = f(x,y)$$

$$F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

$$F_y + F_z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

ex: $y^2 - x^2 - \sin xy = 0 \quad , \quad y = f(x)$

$$\frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{(-2x - (\cos xy)\, y)}{2y - (\cos xy)\, x}$$

ex: $x^3 + z^2 + ye^{xz} + z\cos y = 0$

$z = f(x,y)$

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)} = ?$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2 + ye^{xz} \cdot z)}{2z + ye^{xz} \cdot x + \cos y}$$

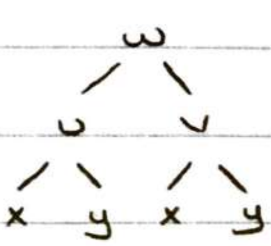
$$\left.\frac{\partial z}{\partial x}\right|_{(0,0,0)} = \frac{0}{1}$$

(book ur)

ex: Show that $w = f(u,v)$ satisfies the Laplace equation

$\boxed{f_{uu} + f_{vv} = 0}$ and if $\boxed{u = \frac{x^2 - y^2}{2}}$ $\boxed{v = xy}$ then w satisfies the Laplace equation

$$\boxed{w_{xx} + w_{yy} = 0}$$



$u_x = x$

$u_y = -y$

$v_x = y$

$v_y = x$

$$\underline{w_x} = w_u u_x + w_v v_x$$

$$= w_u x + w_v y$$

$$\underline{(w_u)_x} = (w_u)_u u_x + (w_u)_v \cdot v_x$$

$$= w_{uu} u_x + w_{uv} v_x$$

$$= w_{uu} x + w_{uv} y$$

$$\underline{(w_v)_x} = w_{vu} u_x + w_{vv} v_x$$

$$= w_{vu} \cdot x + w_{vv} \cdot y$$

$$\underline{w_{xx}} = (w_u)_x x + w_u \cdot 1 + (w_v)_x y$$

$$= (w_{uu} x + w_{uv} y) x + w_u + (w_{vu} x + w_{vv} y)$$

$$\underline{w_y} = w_u u_y + w_v v_y$$

$$= w_u (-y) + w_v x$$

$$\underline{w_{yy}} = (w_{uu} u_y + w_{uv} v_y)(-y) + w_u(-1) + (w_{vu} u_y + w_{vv} v_y) x$$

$$= (w_{uu} \cdot (-y) + w_{uv} x)(-y) - w_u + (w_{vu}(-y) + w_{vv} x) x$$

$$w_{xx} + w_{yy} = w_{uu}(x^2+y^2) + w_{uv}(xy-xy) + w_{vu}(xy-xy) + w_{vv}(y^2+x^2$$

$$= (w_{uu} + w_{vv})(x^2+y^2)$$

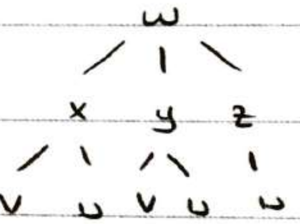
$$= \underbrace{(f_{uu} + f_{vv})}_{=0} (x^2+y^2)$$

$$= 0 \Rightarrow w_{xx} + w_{yy} = 0$$

(book 341)

ex: Find $\partial w / \partial v$ when $u=-1$, $v=2$ if

$w = xy + \ln z$, $x = v^2/u$, $y = u+v$, $z = \cos u$



$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= y \cdot \frac{2v}{u} + x \cdot 1$$

$u=-1$, $v=2 \Rightarrow x = 4/-1 = -4$

$y = -1+2 = 1$

$$\left.\frac{\partial w}{\partial v}\right|_{\substack{u=-1 \\ v=2}} = 1 \cdot \frac{2 \cdot 2}{-1} + (-4) \cdot 1 = -8$$

14.5 → DIRECTIONAL DERIVATIVES AND GRADIENTS

$u = u_1 i + u_2 j$ unit vector

$f : \mathbb{R}^2 \to \mathbb{R}$

$$D_{\vec{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

↓

directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$

ex: Using the definition, find the derivative of

$f(x,y) = x^2 + xy$ at $\overset{x_0\ y_0}{(1,2)}$ in the direction

$$\vec{u} = \underbrace{\frac{\sqrt{2}}{2}}_{u_1} i + \underbrace{\frac{\sqrt{2}}{2}}_{u_2} j$$

$$D_{\vec{u}} f(1,2) = \lim_{t \to 0} \frac{f\left(1 + t \cdot \sqrt{2}/2,\ 2 + t \cdot \sqrt{2}/2\right) - f(1,2)}{t}$$

$$= \lim_{t \to 0} \frac{\left(1 + t\frac{\sqrt{2}}{2}\right)^2 + \left(1 + t\frac{\sqrt{2}}{2}\right)\left(2 + t\frac{\sqrt{2}}{2}\right) - (1^2 + 1 \cdot 2)}{t}$$

$$= \lim_{t \to 0} \frac{\cancel{1} + \sqrt{2}t + t^2/_2 + \cancel{2} + \frac{3}{2}\sqrt{2}t + t^2/_2 - \cancel{3}}{t}$$

$$= \lim_{t \to 0} \sqrt{2} + \cancel{t/_2} + 3\sqrt{2}/_2 + \cancel{t/_2}$$

$$= \frac{5\sqrt{2}}{2}$$

** *NOTE:

$$\frac{\partial f}{\partial x}(x_0, y_0) = D_{\vec{u}} f(x_0, y_0)$$
$\downarrow$ $i + 0j$

$$\frac{\partial f}{\partial y}(x_0, y_0) = D_{\vec{u}} f(x_0, y_0)$$
$\downarrow$ $0i + j$

$$D_{\vec{u}} f(x_0, y_0) = \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2) \Big|_{t=0}$$

$$= f_x(x_0 + tu_1, y_0 + tu_2)\, u_1 + f_y(x_0 + tu_1, y_0 + tu_2)\, u_2$$

$$= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2$$

$$= [f_x(x_0, y_0)\, i + f_y(x_0, y_0)\, j] \cdot [u_1 i + u_2 j]$$

$\downarrow$ definition

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0)\, \vec{u}}$$

$\hookrightarrow$ Gradient vector of f.

$$\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\, i + \frac{\partial f}{\partial y}(x_0, y_0)\, j$$

ex: Previous example revisited.

$$f = x^2 + xy \Rightarrow \nabla f = (2x+y)i + (x)j$$

$$\nabla f(1,2) = 4i + j$$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = (4i+j) \cdot \left(\sqrt{2}/_2 i + \sqrt{2}/_2 j\right)$$

$$= \frac{4}{2}\sqrt{2} + 1 \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{5\sqrt{2}}{2}$$

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \vec{u}$$

unit vector

$$= |\nabla f(x_0, y_0)| |\vec{u}| \cos\theta$$

↳ angle between ∇f and $\vec{u}$

$$= |\nabla f(x_0, y_0)| \cos\theta$$

* if $\cos\theta = 1 \Rightarrow D_{\vec{u}} f(x_0, y_0)$ is max
  $\Updownarrow$
  $\theta = 0 \Rightarrow \vec{u}$ and ∇f are in the same direction

A function most rapidly increases in the direction of its gradient vector

* if $\cos\theta = -1 \Rightarrow D_{\vec{u}} f(x_0, y_0)$ is min
  $\Updownarrow$
  $\theta = \pi \Rightarrow \vec{u}$ and ∇f are in opposite direction

* if $\cos\theta = 0 \Rightarrow D_{\vec{u}} f(x_0, y_0) = 0$
  $\Updownarrow$
  $\theta = \pi/2 \Rightarrow \vec{u}$ and ∇f are perpendicular

! A function most rapidly decreases in the opposite direction of its gradient vector.

! A function is constant in the direction perpendicular to its gradient vector.

ex: Find the directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

a) Increases most rapidly at the point $(1,1)$

$\nabla f = x^2 i + y^2 j$

$\nabla f(1,1) = i + j$

$\vec{u} = \frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{i+j}{\sqrt{1^2+1^2}} = \frac{i+j}{\sqrt{2}} = \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

b) Decreases most rapidly at $(1,1)$

$\vec{u} = -\frac{i}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

c) Stays constant at $(1,1)$

unit vector

$\vec{u} = u_1 i + u_2 j \quad , \quad u_1^2 + u_2^2 = 1$

$\vec{u} \cdot \nabla f = 0 \qquad \nabla f = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

$u_1 \cdot \frac{1}{\sqrt{2}} + u_2 \cdot \frac{1}{\sqrt{2}} = 0$

$u_1 + u_2 = 0$

$u_1 = -u_2$

$u_1^2 + |-u_1|^2 = 1 \quad = u_1^2 = 1/2$

$u_1 = \mp 1/2$

Two direction $\vec{u} = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$

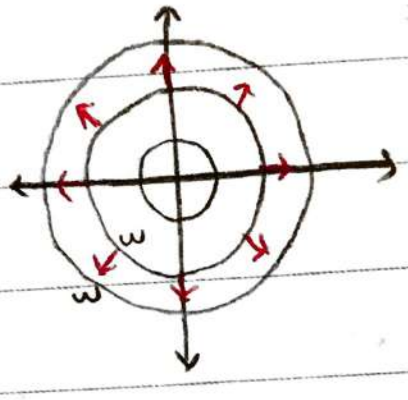
$\vec{u} = -\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

Gradients and Tangents to Level Curves

ex: $f(x,y) = x^2+y^2$

level curves

$x^2+y^2 = f(x,y) = \text{constant} = c$



$\nabla f = 2x\hat{i} + 2y\hat{j}$

In general,

$f(x,y)=c$ level curve.

Parametrization of level curve

$\vec{r}(t) = g(t)\,i + h(t)\,j$

This means

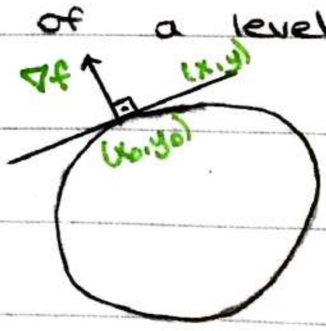
$f(g(t), h(t)) = \text{constant}$

$\frac{d}{dt} f(g(t), h(t)) = 0$

$f_x\, g'(t) + f_y\, h'(t) = 0$

$\nabla f \cdot \underbrace{\frac{d\vec{r}}{dt}}_{\text{tangent to level curve}} = 0$

Gradient vector is perpendicular to the tangent vector of a level curve.



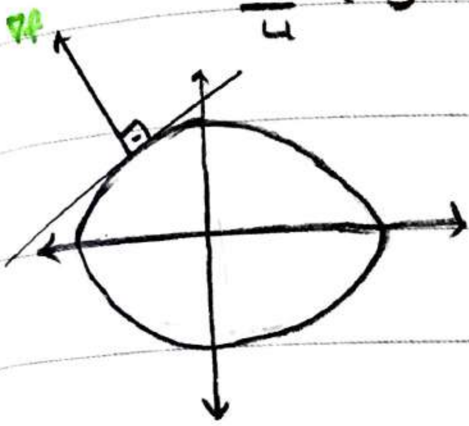
$[(x-x_0)i + (y-y_0)j] \cdot \nabla f = 0$

$$\boxed{\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) = 0}$$

tangent line of a level curve passing from (x_0, y_0)

ex: Find an equation for the tangent line to the ellipse

$\frac{x^2}{4} + y^2 = 2$ at the point $(-2, 1)$



$f = \frac{x^2}{4} + y^2$

level curve of f.

$\nabla f = \frac{x}{2} i + 2y j$

$\nabla f(-2, 1) = -i + 2j$

$\nabla f \, ((x+2) i + (y-1) j) = 0$

$(-x+2) + 2(y-1) = 0$ equation for the tangent line of the level curve

(book 16)

ex: $f(x, y, z) = x^2 + 2y^2 - 3z^2$

$\vec{u} = i + j + k$

Find the derivative of f in the direction of $\vec{u}$ at the point $(1, 1, 1)$.

Attention : $\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{i + j + k}{\sqrt{1+1+1}} = \frac{i}{\sqrt{3}} + \frac{j}{\sqrt{3}} + \frac{k}{\sqrt{3}}$

$D_{\vec{v}} f(1,1,1) = \nabla f(1,1,1) \cdot \vec{v}$

$\nabla f(x, y, z) = 2x i + 4y j - 6z k$

$\nabla f(1,1,1) = 2i + 4j - 6k$

$D_{\vec{v}} f(1,1,1) = (2i + 4j - 6k) \cdot \left(\frac{i}{\sqrt{3}} + \frac{j}{\sqrt{3}} + \frac{k}{\sqrt{3}} \right)$

$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{6}{\sqrt{3}} = 0$

(book 32)

ex: In what direction is the derivative of

$$f(x,y) = \frac{(x^2-y^2)}{x^2+y^2} \quad \text{at } (1,1) \text{ equal to zero?}$$

$$\nabla f = \left[\frac{2x\,(x^2+y^2) - (x^2-y^2)\cdot 2x}{(x^2+y^2)^2}\right]\vec{i} + \left[\frac{-2y(x^2+y^2) - (x^2-y^2)\cdot 2y}{(x^2+y^2)^2}\right]\vec{j}$$

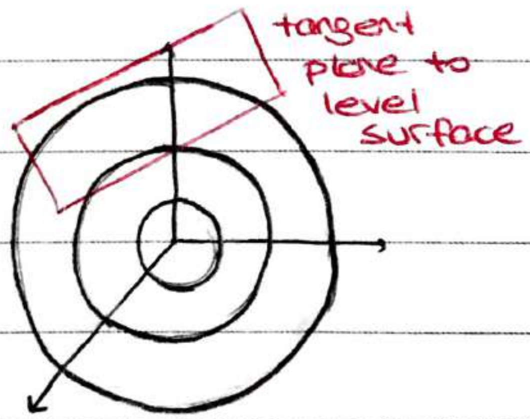
$$\nabla f(1,1) = \frac{4}{4}\,\vec{i} - \frac{4}{4}\,\vec{j} = \vec{i} - \vec{j}$$

14.6 → TANGENT PLANES

$f(x,y,z) = \text{constant} \Rightarrow$ level surface of f

ex: $f(x,y,z) = x^2+y^2+z^2$

$$x^2+y^2+z^2 = c$$



∇f is the normal vector of the tangent plane to the level surface

$$\nabla f \cdot \left((x-x_0)\vec{i} + (y-y_0)\vec{j} + (z-z_0)k\right) = 0$$

equation of the tangent plane

Equation of the tangent plane

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,z_0)(y-y_0)+\frac{\partial f}{\partial z}(x_0,y_0,z_0)(z-z_0)=0$$

Equation of normal line

$$x = x_0 + \frac{\partial f}{\partial x}(x_0,y_0,z_0)\,t$$

$$y = y_0 + \frac{\partial f}{\partial y}(x_0,y_0,z_0)\,t$$

$$z = z_0 + \frac{\partial f}{\partial z}(x_0,y_0,z_0)\,t$$

$$\vec{r}(t) = (x_0 i + y_0 j + z_0 k) + t\nabla f$$

ex: Find the tangent plane and normal line of the level surface

$f(x,y,z) = x^2+y^2+z-9 = 0$ at the point $P(1,2,4)$

$\nabla f = 2x i + 2y j + k$

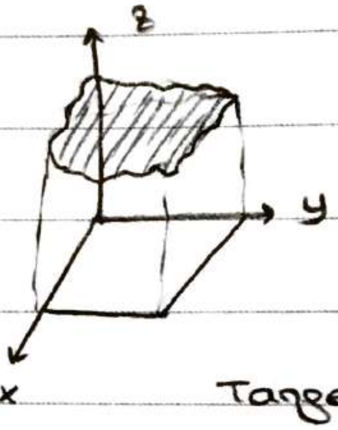
Tangent plane: $2(x-1)+4(y-2)+(z-4)=0$

Normal Line: $x = 1+2t$

$y = 2+4t$

$z = 4+t$

ex: Find an equation for the tangent plane to the graph of $z=f(x,y)$



$$F(x,y,z) = f(x,y) - z$$

$$F=0 \iff z=f(x,y)$$

Tangent plane to $F(x,y,z)=0$

$$0 = \frac{\partial F}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial F}{\partial z}(x_0,y_0,z_0)(z-z_0)$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} \qquad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} \qquad \frac{\partial F}{\partial z} = -1$$

$$\overbrace{\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) - (z-z_0) = 0}^{\text{Tangent Plane}}$$

ex: Find the plane tangent to the surface

$$z = \underbrace{x\cos y - ye^x}_{f(x,y)} \quad \text{at} \quad (\underset{x_0}{0},\underset{y_0}{0},\underset{z_0}{0})$$

$$\frac{\partial f}{\partial x} = \cos y - ye^x \qquad \frac{\partial f}{\partial x}(0,0) = 1-0 = 1$$

$$\frac{\partial f}{\partial y} = x\sin y - e^x \qquad \frac{\partial f}{\partial y}(0,0) = 0-1 = -1$$

$$1(x-0) - 1(y-0) - (z-0) = 0$$

$$\boxed{x-y-z=0} \longrightarrow \text{equation of tangent plane at } (0,0,0)$$